

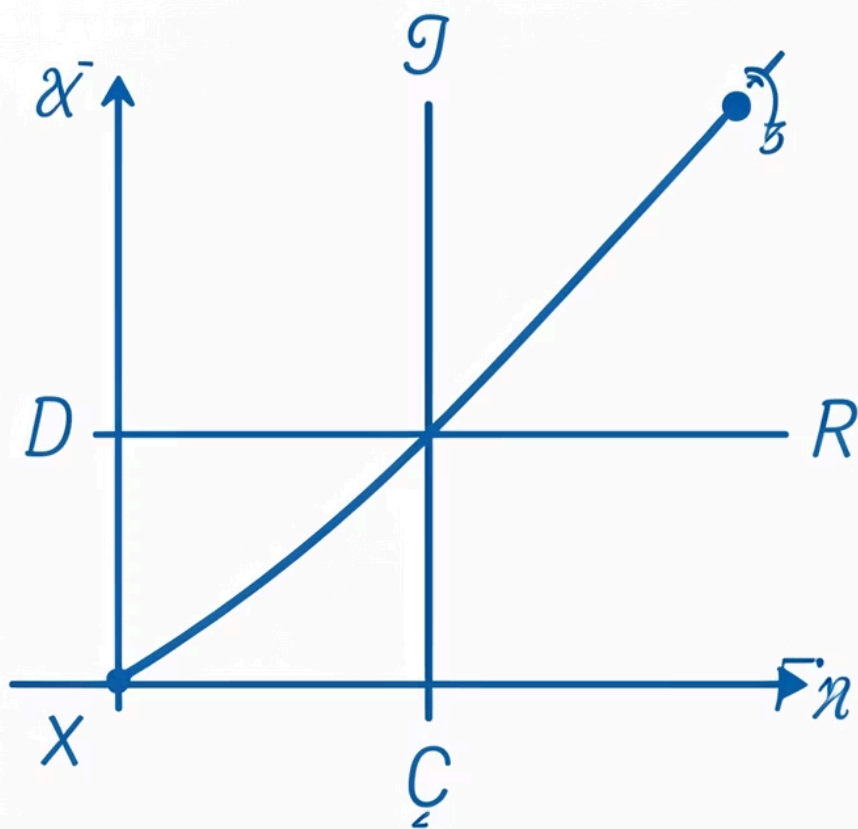
Bong Study Hub

Applications of Derivatives

Mastering WBJEE & JEE Mains Calculus: A Comprehensive Guide



Rate of Measure



Understanding Change

Derivatives measure how one quantity changes with respect to another. In JEE Mains, this concept is crucial for solving problems involving **related rates**.

The derivative dy/dx represents the instantaneous rate of change of y with respect to x .

Step 1

Identify the relationship between variables using given equations

Step 2

Differentiate implicitly with respect to time or another variable

Step 3

Substitute known values and solve for the unknown rate

Derivative as a Rate of Change

The derivative is the mathematical tool used to measure how one quantity changes with respect to another. In the context of calculus for JEE Mains, it quantifies the sensitivity of a function's output to changes in its input.

Average Rate of Change

The average rate of change over an interval is defined as:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Here, Δx represents the change in input, and Δy represents the corresponding change in output. It measures the net change per unit over a specific interval.

From Average to Instantaneous

To find the **instantaneous** rate of change, we shrink the interval Δx towards zero. This transition is governed by the limit concept:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

As Δx becomes infinitesimally small, the average rate approaches the precise rate at a single point.

Definition of Derivative

The derivative is formally defined as the limit of the average rate of change as the interval length approaches zero:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This expression gives us the instantaneous rate of change of y with respect to x at any specific value of x .

Interpretation and Geometric Meaning

Beyond its physical interpretation as a velocity or growth rate, the derivative **dy/dx** has a clear geometric meaning: it represents the **slope of the tangent line** to the curve at point x . It tells us exactly how steep the function is at that specific moment.

- ❑ **Key Takeaway:** While the *average* rate of change provides the slope of a secant line connecting two points on a graph, the *instantaneous* rate of change (derivative) provides the slope of the tangent line at a single point, offering a precise look at local behavior.

JEE Main 2020 ,3 Sept-II- Rate of Change Problem

❏ **Problem Statement:** If the surface area of a cube is increasing at a rate of $3.6 \text{ cm}^2/\text{sec}$ while retaining its shape, find the rate of change of its volume (in cm^3/sec) when the length of a side of the cube is 10 cm .

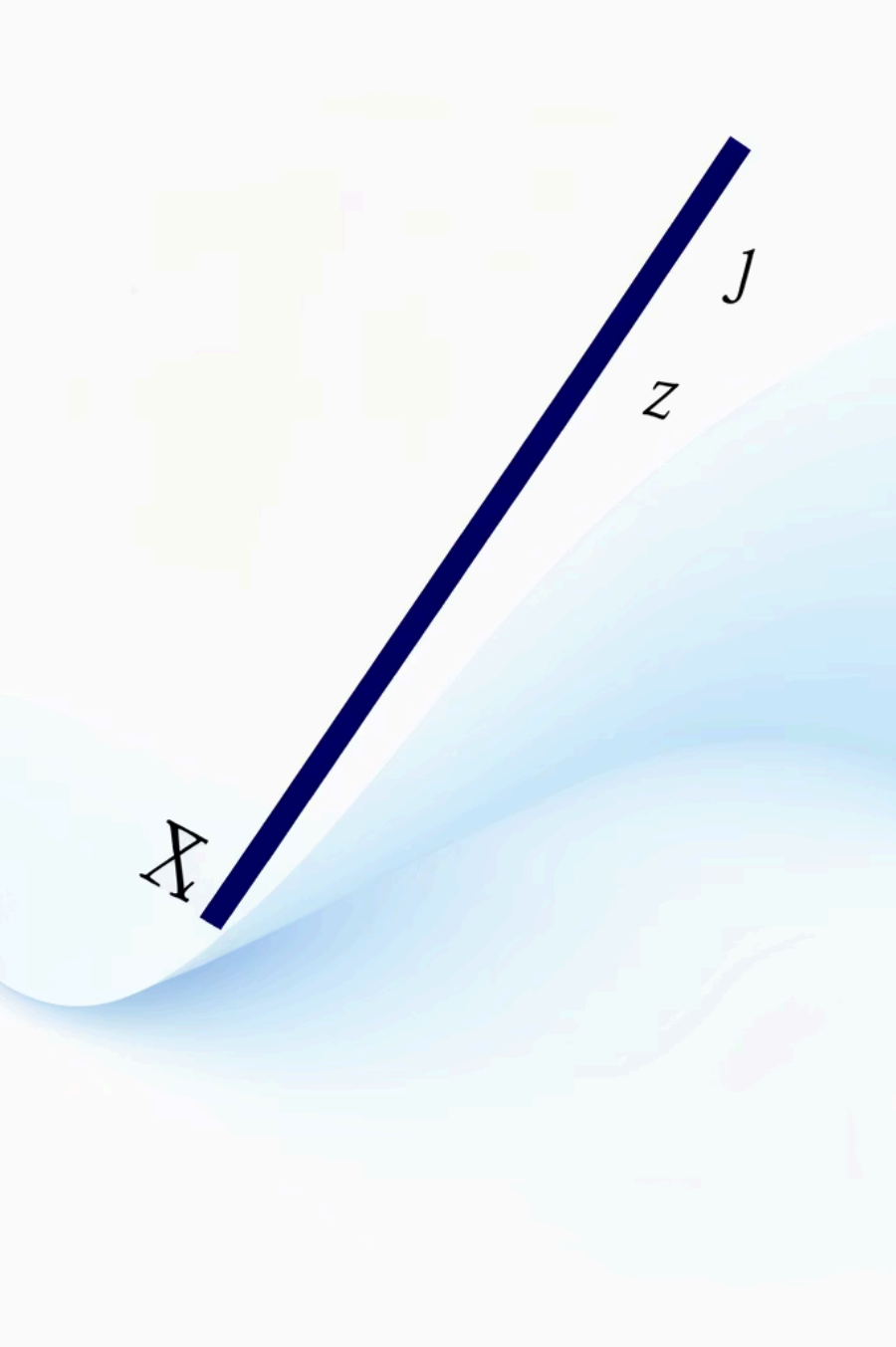
Question: What is the rate of change of volume at that moment?

A) 20

B) 10

C) 18

D) 9



Tangents and Normals

The derivative $f'(x)$ gives the slope of the tangent line to the curve $y = f(x)$ at any point.

Tangent Line

Equation: $y - y_1 = m(x - x_1)$, where $m = f'(x_1)$

Normal Line

Perpendicular to tangent, slope = $-1/f'(x_1)$

Key Point

At points where $f'(x) = 0$, tangent is horizontal; where $f'(x)$ is undefined, tangent is vertical

What are Tangent and Normal?

In calculus, we study the behavior of curves and aim to understand their direction and rate of change at specific, individual points.

Two Important Lines

Tangent line: A line that touches the curve at a single point, representing the instantaneous direction or slope of the curve at that location.

Normal line: A line that is perpendicular to the tangent line at the point of contact, representing the direction of the steepest ascent or descent perpendicular to the curve.



Tangent to a Curve

A **tangent line** is a straight line that touches a curve at a single, specific point. At this point, the tangent line has the exact same slope or instantaneous direction as the curve itself. It gives us a precise understanding of the curve's behavior at that particular location.

Key Idea: The Derivative as Slope

One of the most fundamental applications of derivatives in calculus is that the slope of the tangent line to a function's graph at any given point is equal to the value of the derivative of the function at that same point.

- Slope of the tangent line = $\frac{dy}{dx}$

Equation of the Tangent Line

If a curve is given by $y = f(x)$, and we want to find the equation of the tangent line at a point (x_1, y_1) on the curve, we use the point-slope form of a linear equation:

$$y - y_1 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} (x - x_1)$$

Here, $\left(\frac{dy}{dx} \right)_{(x_1, y_1)}$ represents the derivative of the function evaluated at the point (x_1, y_1) , which gives the slope of the tangent at that specific point.

Normal to a Curve

In calculus, the normal line is intrinsically linked to the tangent line at a specific point on a curve. It provides crucial information about the curve's behavior, particularly its orientation perpendicular to the direction of motion.

Definition

A **normal line** is a straight line that passes through the point of contact on a curve and is strictly perpendicular to the tangent line at that exact same point.

Key Idea: Relationship to Tangent Slope

Because the normal line is perpendicular to the tangent line, its slope is the negative reciprocal of the slope of the tangent. This inverse relationship is fundamental for determining its equation.

Formulas for the Normal Line

- Slope of the normal line: $m_{\text{normal}} = -\frac{1}{\left(\frac{dy}{dx}\right)}$ (provided $\frac{dy}{dx} \neq 0$)
- Equation of the normal line at a point (x_1, y_1) on the curve:

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

Here, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the slope of the tangent line evaluated at the point (x_1, y_1) .

Geometric Relationship: Tangent and Normal

Understanding how tangent and normal lines interact with a curve is fundamental to advanced calculus problems.

01

Point of Contact

Both the tangent line and the normal line pass through the exact same point on the curve, which is known as the point of contact.

02

Tangent's Role

The tangent line touches the curve at this single point, providing the instantaneous slope or direction of the curve.

03

Normal's Perpendicularity

The normal line is always perpendicular to the tangent line at the point of contact, forming a 90-degree angle between them.

Mathematical Relation

This perpendicular relationship has a direct consequence for their slopes:

$$m_{\text{tangent}} \times m_{\text{normal}} = -1$$

Where m represents the slope of the respective line.

This visual emphasizes the critical perpendicular relationship between the tangent and normal lines at any given point on a curve.

Example: Tangent and Normal to $y = x^2$

1. The Curve and its Derivative

Consider the curve defined by:

$$y = x^2$$

Its derivative, representing the general slope of the tangent at any point x , is:

$$\frac{dy}{dx} = 2x$$

2. At Point $x = 1$

We want to find the tangent and normal lines at $x = 1$.

- Point on curve: When $x = 1$, $y = 1^2 = 1$. So the point is $(1, 1)$.
- Slope of tangent: Substitute $x = 1$ into the derivative: $m_{\text{tangent}} = 2(1) = 2$.

3. Equation of the Tangent Line

Using the point-slope form $y - y_1 = m(x - x_1)$:

$$y - 1 = 2(x - 1)$$

Simplified: $y = 2x - 1$

4. Equation of the Normal Line

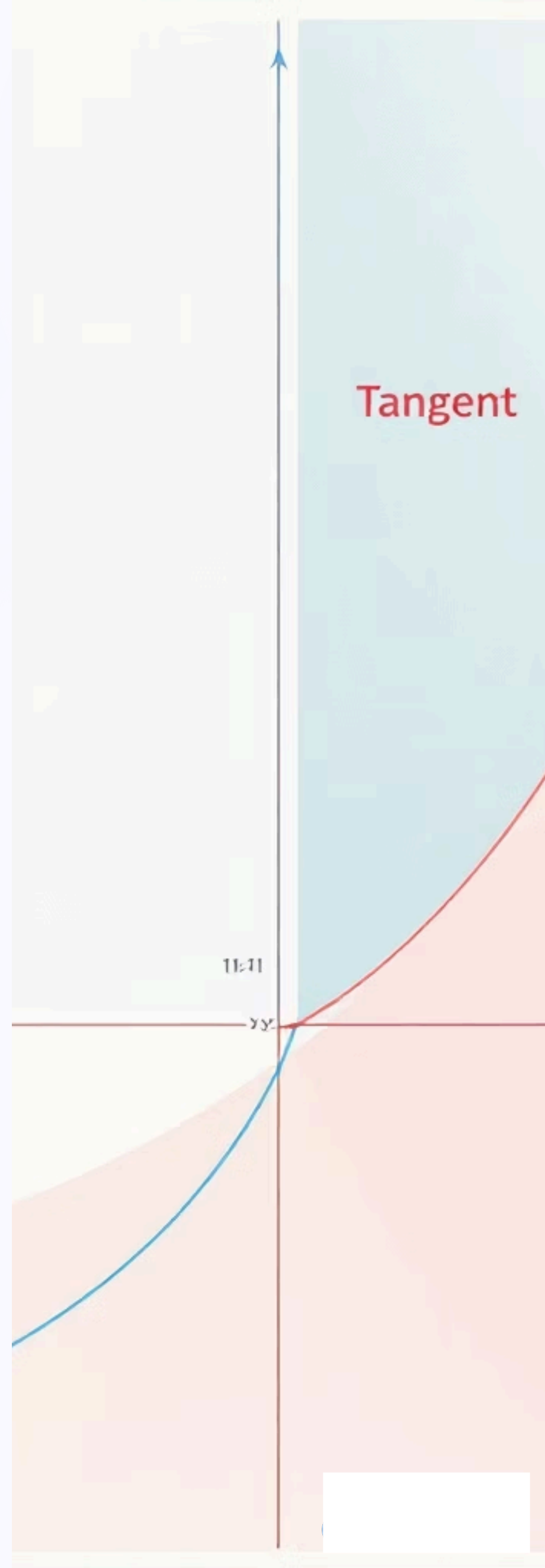
The normal line is perpendicular to the tangent, so its slope is the negative reciprocal of the tangent's slope:

$$m_{\text{normal}} = -1/2$$

Using the point-slope form $y - y_1 = m(x - x_1)$:

$$y - 1 = -\frac{1}{2}(x - 1)$$

Simplified: $y = -\frac{1}{2}x + \frac{3}{2}$



Summary: Tangent and Normal Lines

A concise overview of the fundamental properties and equations for tangent and normal lines to a curve, crucial for calculus applications and competitive exams.

Tangent Line

- Touches the curve at a single point.
- Represents the instantaneous direction of the curve.
- Slope is given by the first derivative: $\frac{dy}{dx}$.

Equation at (x_1, y_1) :

$$y - y_1 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} (x - x_1)$$

Normal Line

- Perpendicular to the tangent line at the point of contact.
- Slope is the negative reciprocal of the tangent's slope: $-\frac{1}{\left(\frac{dy}{dx}\right)}$.

Equation at (x_1, y_1) :

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx} \right)_{(x_1, y_1)}} (x - x_1)$$

Key Takeaway

Tangent and normal lines are fundamental concepts in calculus, providing insight into the local behavior and geometry of curves. They are essential tools for curve analysis in advanced mathematics and problem-solving in exams like JEE Mains.

WBJEE 2023 (MCQ)

- ❑ **Problem Statement:** The tangent at the point $(a \cos \theta, b \sin \theta)$, where $0 < \theta < \pi/2$, to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the x-axis at point T and the y-axis at point T₁. Let O be the origin. Find the minimum value of $(OT)(OT_1)$ for $0 < \theta < \pi/2$.

Key Information

Ellipse Equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Point of Contact: $(a \cos \theta, b \sin \theta)$

Goal: Find the minimum value of $(OT)(OT_1)$

Multiple Choice Options

A) ab

B) $2ab$

C) 0

D) 1

Angle of Intersection & Lengths

Angle of Intersection

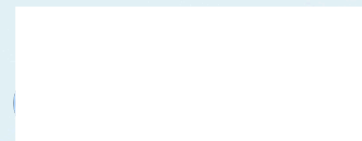
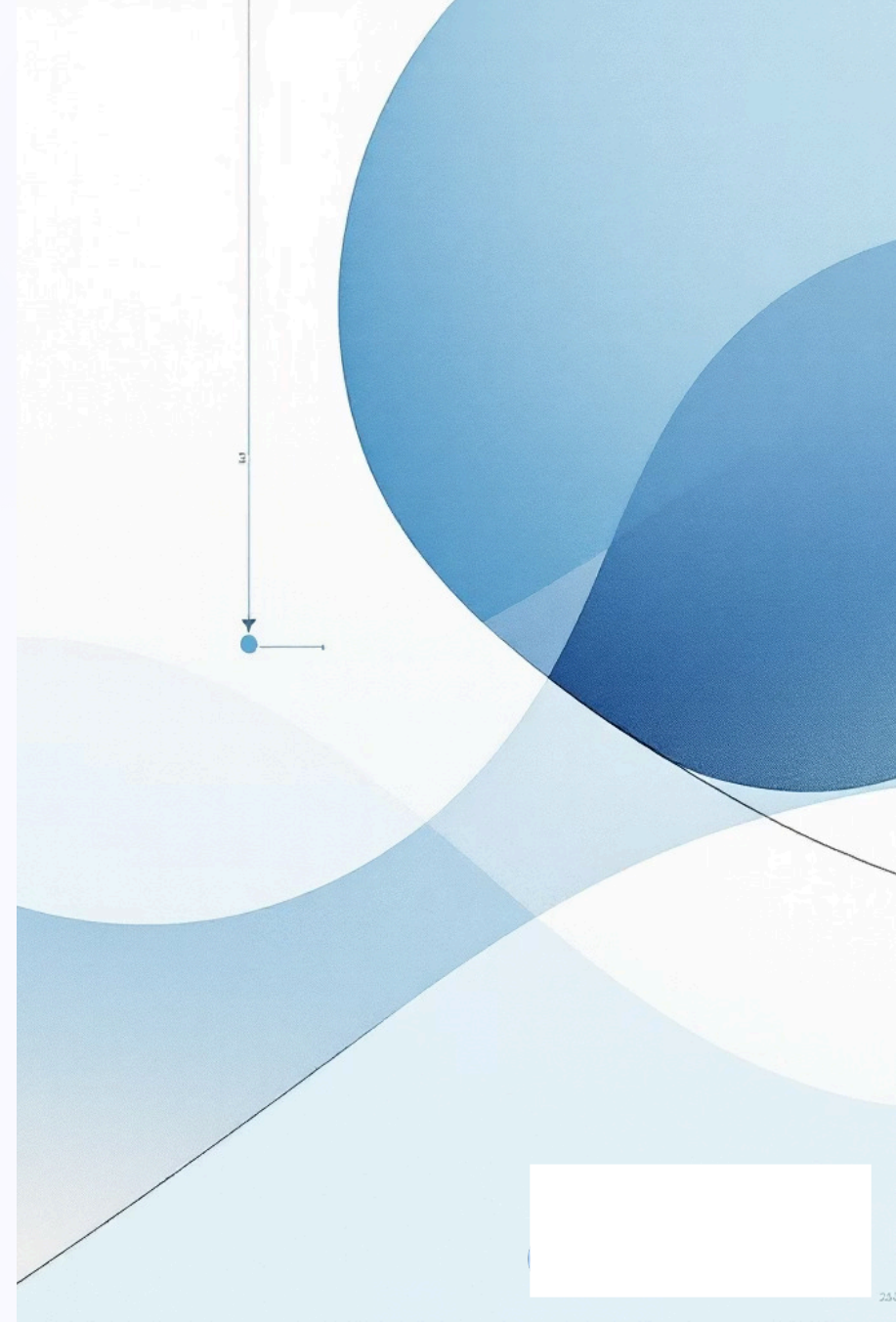
For two curves $y = f(x)$ and $y = g(x)$, the angle θ between them is:

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

where $m_1 = f'(x)$ and $m_2 = g'(x)$

Length Calculations

- **Length of Tangent:** $|y\sqrt{1 + m^2}|/m$
- **Length of Normal:** $|y\sqrt{1 + m^2}|$
- **Sub-tangent:** $|y/m|$
- **Sub-normal:** $|ym|$



Rolle's Theorem & LMVT



Rolle's Theorem

If $f(x)$ is continuous on $[a,b]$, differentiable on (a,b) , and $f(a) = f(b)$, then there exists at least one $c \in (a,b)$ such that $f'(c) = 0$



Lagrange's Mean Value Theorem

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) , then there exists at least one $c \in (a,b)$ such that $f'(c) = [f(b) - f(a)] / (b - a)$

- ❏ **Geometric Interpretation:** LMVT guarantees that for any curve, there's at least one point where the tangent is parallel to the secant joining the endpoints.

Intermediate Value Theorem & Critical Points

Intermediate Value Theorem (IVT)

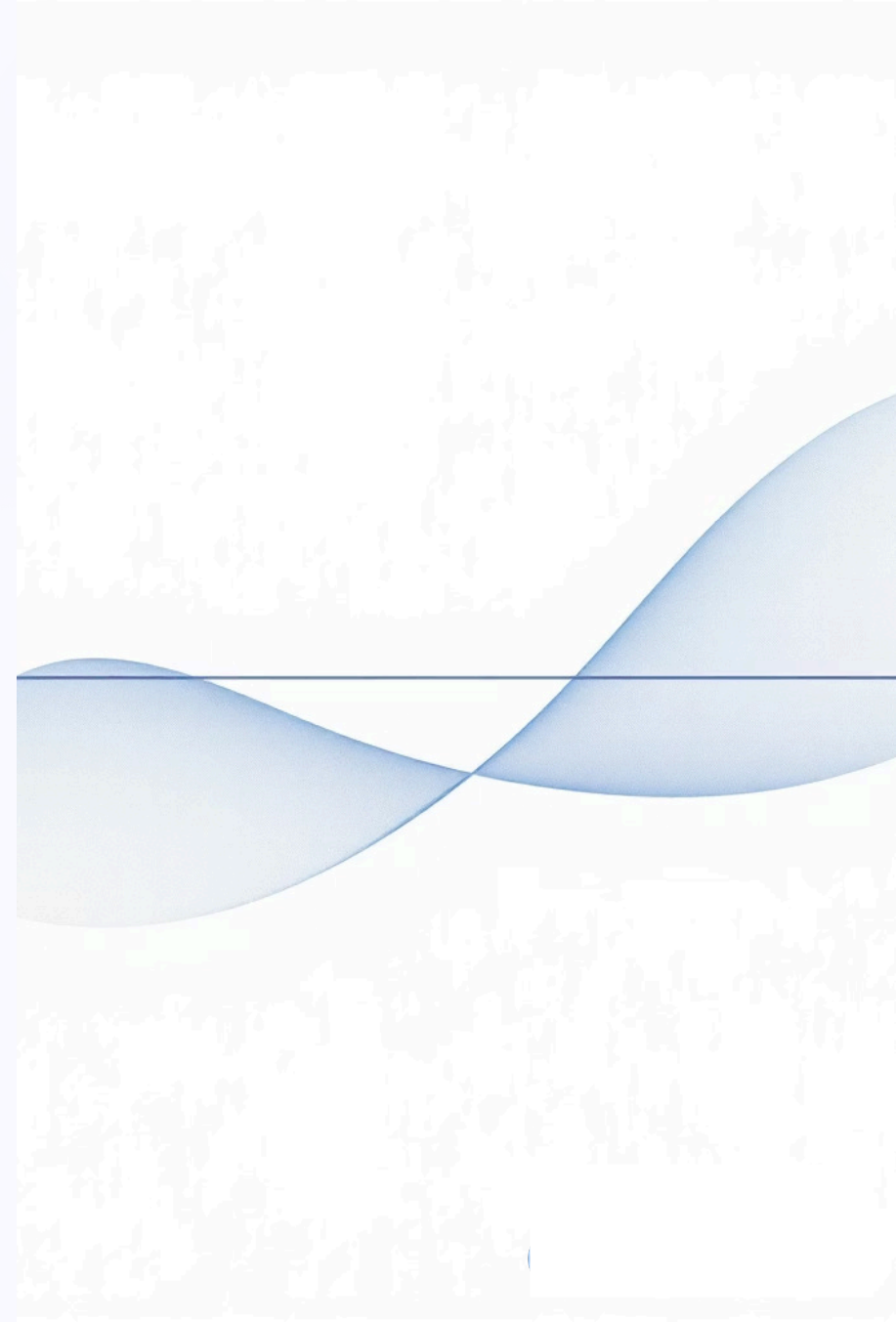
If $f(x)$ is continuous on $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there exists at least one $c \in (a,b)$ such that $f(c) = k$.

JEE Application: Proving existence of roots and solutions in specific intervals.

Critical Points

Points where $f'(x) = 0$ or $f'(x)$ is **undefined** are called **critical points**.

These points are potential locations for local maxima, minima, or points of inflection.



Increasing & Decreasing Functions



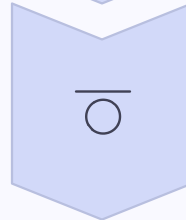
Increasing

$f'(x) > 0 \Rightarrow$ function is increasing



Decreasing

$f'(x) < 0 \Rightarrow$ function is decreasing



Constant

$f'(x) = c \Rightarrow$ function is constant

Monotonic Functions

A function is **monotonic** if it's either entirely non-increasing or non-decreasing in an interval.

For composite functions $f(g(x))$, apply chain rule: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

JEE Main 2021

- **Problem Statement:** Let R be the least value of a such that the function $f(x) = x^2 + ax + 1$ is **increasing** on the interval $[1,2]$. Let S be the greatest value of a such that the function $f(x) = x^2 + ax + 1$ is **decreasing** on the interval $[1,2]$.

Key Information

To solve this, we first find the derivative of the function:

$$f'(x) = 2x + a$$

For Increasing Function

$$f'(x) \geq 0 \text{ on } [1,2]$$

Since $f'(x)$ is an increasing linear function, its minimum value on $[1,2]$ occurs at $x = 1$.

$$2(1) + a \geq 0 \Rightarrow 2 + a \geq 0 \Rightarrow a \geq -2$$

The least value of a is $R = -2$.

For Decreasing Function

$$f'(x) \leq 0 \text{ on } [1,2]$$

Since $f'(x)$ is an increasing linear function, its maximum value on $[1,2]$ occurs at $x = 2$.

$$2(2) + a \leq 0 \Rightarrow 4 + a \leq 0 \Rightarrow a \leq -4$$

The greatest value of a is $S = -4$.

Find the value of $|R - S|$

Substituting the values of R and S :

$$|R - S| = |-2 - (-4)| = |-2 + 4| = |2| = 2$$

The final value is **2**.

Maxima & Minima

01

Find Critical Points

Solve $f'(x) = 0$ or where $f'(x)$ is undefined

02

First Derivative Test

Check sign change of $f'(x)$ around critical points

03

Classify Extrema

Local max/min based on sign changes

04

Compare Values

For global extrema, compare values at critical points and endpoints

Local vs Global: Local extrema are highest/lowest in a neighborhood; global extrema are highest/lowest in the entire domain.



Second Derivative Test & Concavity

Second Derivative Test

If $f'(c) = 0$ and $f''(c) < 0 \Rightarrow$ local maximum

If $f'(c) = 0$ and $f''(c) > 0 \Rightarrow$ local minimum

If $f''(c) = 0 \Rightarrow$ test is inconclusive

Concavity

$f''(x) > 0 \Rightarrow$ concave up (smiling curve)

$f''(x) < 0 \Rightarrow$ concave down (frowning curve)

JEE MAINS 2025

Let the function be defined as:

$$f(x) = \int_0^{x^2} \frac{t^2 - 8t + 15}{e^t} dt, \quad \text{where } x \in \mathbb{R}$$

Question :

Find the number of **local maximum points** and **local minimum points** of the function $f(x)$.

A) 3 and 2

C) 2 and 3

B) 2 and 2

D) 1 and 3

JEE Mains April 2025

Problem Statement

If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its local maximum and local minimum values at p and q , respectively, such that $p^2 = q$:

Key Information:

- Function: $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$
- Condition: $a > 0$
- Local maximum at: $x = p$
- Local minimum at: $x = q$
- Relationship: $p^2 = q$

Question

Find the value of $f(3)$.

A) 10

B) 55

C) 23

D) 37

WBJEE 2023 (MCQ)

Problem Statement

A missile is fired from ground level and rises vertically upward. The height x meters reached after t seconds is given by:

$$x = 100t - \frac{25}{2}t^2$$

Key Information:

- Initial position: ground level
- Height function: $x = 100t - \frac{25}{2}t^2$
- Variable t : time in seconds
- Variable x : height in meters

Question

Find the **maximum height** reached by the missile.

A) 100 m

C) 200 m

B) 300 m

D) 125 m

WBJEE 2025 (MSQ)

Problem Statement

Let $f(x) = x^3$, where $x \in [-1, 1]$.

Which of the following statements are correct?

□ Key Information:

- Function: $f(x) = x^3$
- Domain: $[-1, 1]$
- Focus: Properties of the derivative $f'(x)$

Options

- (A) f' has a minimum at $x = 0$
- (B) f' has a maximum at $x = 1$
- (C) f' is continuous on $[-1, 1]$
- (D) f' is bounded on $[-1, 1]$

Point of Inflection



Definition

A point where the curve changes its concavity is called a **point of inflection**.

Conditions

- $f''(x) = 0$ or $f''(x)$ is undefined
- $f''(x)$ changes sign around the point
- $f(x)$ is continuous at that point

JEE Tip: Not all points where $f''(x) = 0$ are inflection points—check for sign change!



Onward to Excellence

From all of us at **Bong Study Hub**, we extend our sincerest good wishes for your academic journey and future success.



Keep Learning, keep growing, and shine bright!

Your dedication today fuels your success tomorrow. Believe in your potential and never stop chasing your ambitions.